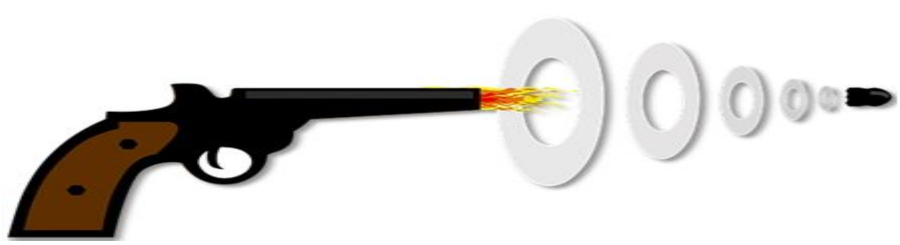
Introduction

Rigid body is a body with a perfectly definite and unchanging shape. The distances between all pairs of particles of such a body do not change.

Bridge can be taken as rigid body as the distance between any two particles does not change or it is so small that it can be neglected

The lemon is not a rigid body since we can see significant change in its shape after application of force

In pure translational motion at any instant of time all particles of the body have the same velocity.

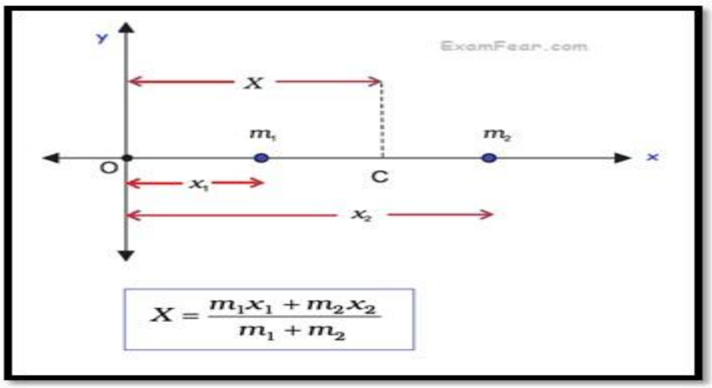


In rotation of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.

The motion of a rigid body which is not pivoted or fixed in some way is either a pure translation or a combination of translation and rotation. The motion of a rigid body which is pivoted or fixed in some way is rotation.


Centre of mass

- Imaginary point where the whole mass of system can be assumed to be concentrated
- The centre of mass of two bodies lies in a straight line.
  (Here m_1 & m_2 are two bodies such that m_1 is at a distance x_1 from O, & m_2 at a distance x_2 from O.)



- The coordinates of centre of mass of a body is given by (X,Y,Z)

$$X = \frac{\sum m_i x_i}{M}$$

$$Y = \frac{\sum m_i y_i}{M}$$

$$Z = \frac{\sum m_i z_i}{M}$$

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Here,

- M = S m_i, the index i runs from 1 to n
- m_i is the mass of the i^{th} particle
- the position of the i^{th} particle is given by (x_i, y_i, z_i).
- If we increase the number of elements n , the element size Dm_i decreases , and coordinates of COM is given by:

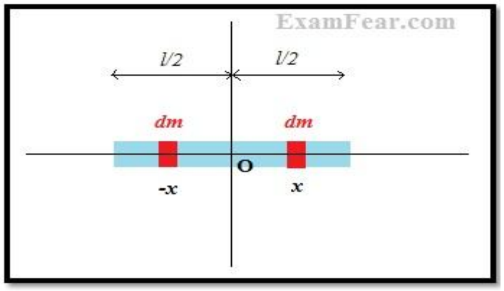
Where x,y,z = coordinates of COM of small element dm

- The vector expression equivalent to these three scalar expressions is

  Where

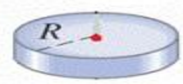
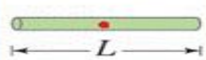
- $\boldsymbol{R}$ = position of COM of body
- $\boldsymbol{r}$ = position of COM of small element of mass dm

Consider a thin rod of length l, taking the origin to be at the geometric centre of the rod and x-axis to be along the length of the rod, we can say that on account of reflection symmetry, for every element dm of the rod at x, there is an element of the same mass dm located at –x.

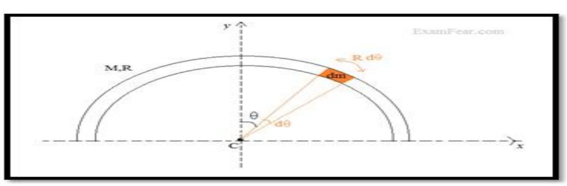


The net contribution of every such pair to the integral and hence the integral x dm itself is zero. Thus the COM coincides with the geometric centre.

- The same symmetry argument will apply to homogeneous rings, discs, spheres, or even thick rods of circular or rectangular cross section; their centre of mass coincides with their geometric centre.

Body	Position of COM	Figure
Thin Circular Ring, Radius R	At center	
Circular disc radius R	At center	
Thin Rod Length L	At half the length	
Solid Sphere radius R	At center	

Example - Find COM of semicircular ring.



Solution: Let us consider a semicircular ring of mass M and radius R.

We know the formula to calculate COM i.e. $X = \frac{1}{M}\int x dm$, $Y = \frac{1}{M}\int y dm$

Now you can see that in x axis, for every small element in $+x$ there is an element at $-x$, so by symmetry these get cancel out, and the COM in x axis lies at the center.

For y axis: For an angle of π, mass of ring – M
then for an angle of dθ, mass of element dm - $\frac{M}{\pi}d\theta$

Also y coordinate of dm is Rcosθ.

So we get $Y = \frac{1}{M}\int y dm = \frac{1}{M}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(R\cos\theta)(\frac{M}{\pi}\,d\theta)$

$Y = 2R/\pi$: y coordinate of COM

Motion of COM

- The centre of mass of a system of particles moves as if all the mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.

$MR = \sum m_i r_i = m_1 r_1 + m_2 r_2 + \cdots + m_n r_n$

Differentiating w.r.t. time

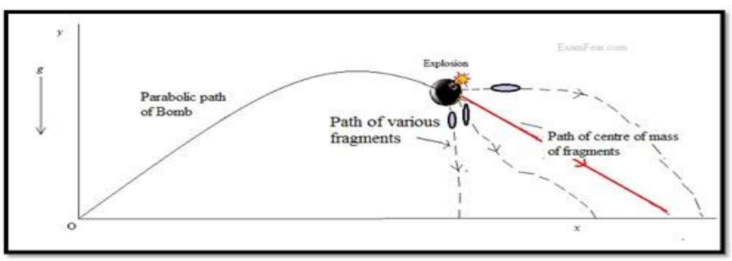
$$M\frac{dR}{dt} = m_1\frac{dr_1}{dt} + m_2\frac{dr_2}{dt} + \cdots + m_n\frac{dr_n}{dt}$$
$$MV = m_1 v_1 + m_2 v_2 + \cdots + m_n v_n$$

Differentiating again w.r.t. time

$$MA = m_1 a_1 + m_2 a_2 + \cdots + m_n a_n$$

- M**A** = $\mathbf{F}_{ext}$ (Since the contribution of internal forces is zero, because they appear in pairs and cancel out each other) Where
- M = S m_i
- **A** = acceleration of COM
- $\mathbf{F}_{ext}$ = sum of all external forces acting on system of particles
- Instead of treating extended bodies as single particles, we can now treat them as systems of particles.

We can obtain the translational component of their motion, i.e. the motion COM of the system, by taking the mass of the whole system to be concentrated at the COM and all the external forces on the system to be acting at the centre of mass.



- When a bomb explodes in a parabolic path, different fragment goes in different path with complex trajectories, but COM continues to travel in the same parabolic path.

Example - A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system?

Solution – In this case if we take (trolley + child) as a system, there is no external force involved.
The force involved in running of child (friction) becomes internal, so the speed of CM of this system remains constant.

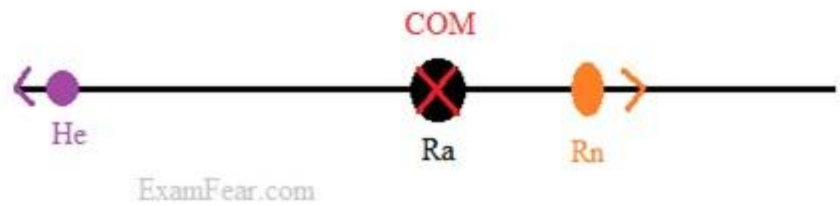
Linear Momentum of System of Particles

- The total momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its COM.
- $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \ldots + \mathbf{p}_n$
  $= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n$
- **P** = M**V**

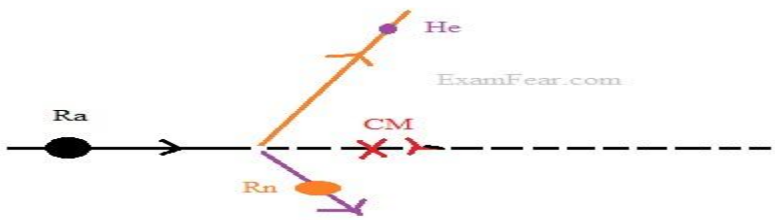
Where

- pi = momentum of i[th] particle ,
- **P** = momentum of system of particles ,
- **V** = velocity of COM

- Newton's Second Law extended to system of particles: d**P**/dt = $\mathbf{F}_{ext}$.
- When the total external force acting on a system of particles is zero ($\mathbf{F}_{ext} = \mathbf{0}$), the total linear momentum of the system is constant (dP/dt = 0 => P = constant). Also the velocity of the centre of mass remains constant (Since P = mv = Constant).
- If the total external force on a body is zero, then internal forces can cause complex trajectories of individual particles but the COM moves with a constant velocity.
- Example: Decay of Ra atom into He atom & Rn Atom
- Case I – If Ra atom was initially at rest, He atom and Rn atom will have opposite direction of velocity, but the COM will remain at rest.



Case II – If Ra atom is having an uniform velocity before , then He and Rn can have complex trajectories but COM will have the same VELOCITY as of Ra atom.



Example - A bullet of mass m is fired at a velocity of v1, and embeds itself in a block of mass M, initially at rest and on a frictionless surface. What is the final velocity of the block?

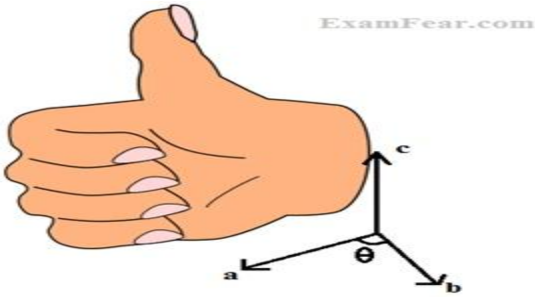
Solution: Now if we take bullet and block as a system, then no external force is acting on it. So we can conserve momentum.

$(m_1\mathbf{v_1} + M\mathbf{v_2})_{\mathbf{before}} = (m_1 + M)\mathbf{v_{after}}$

$\mathbf{v_{after}} = m_1\mathbf{v_1}/(m_1 + M)$, as the initial velocity of block M is zero.

Vector Product

- Vector product (cross product) of two vectors a and b is a × b = ab sinΘ = c , where Θ is angle between a & b
- Vector product c is perpendicular to the plane containing a and b.
- If you keep your palm in direction of vector a and curl your fingers to the direction a to b, your thumb will give you the direction of vector product c



- $a \times b \neq b \times a$
- $a \times b = -b \times a$
- $a \times (b + c) = a \times b + a \times c$
- $a \times a = 0$, 0 is called null vector i.e having zero magnitude
- $\hat{\imath} \times \hat{\jmath} = \hat{k}$
- $\hat{\jmath} \times \hat{k} = \hat{\imath}$
- $\hat{k} \times \hat{\imath} = \hat{\jmath}$

Properties of vector product

$\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ and $\vec{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$ then their vector product is given by

$$\vec{a} \times \vec{b} = \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{matrix}$$

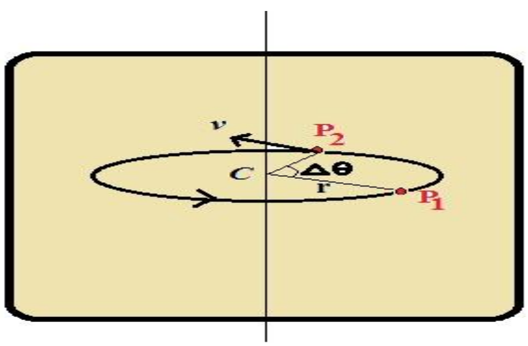
Example - Calculate the cross product between a= (3, −3, 1) and b = (4, 9, 2)

Solution:

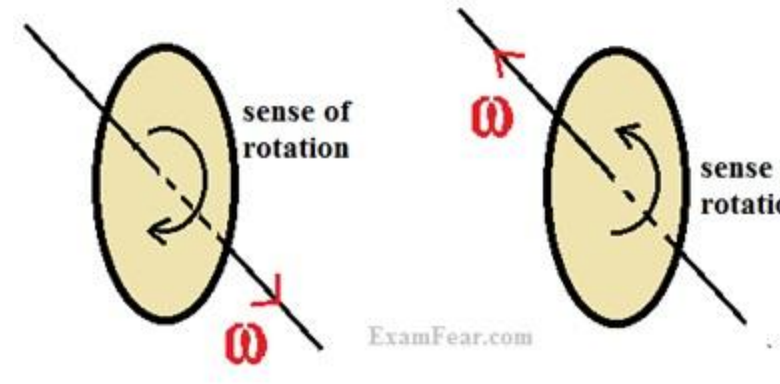
$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 1 \\ 4 & 9 & 2 \end{vmatrix} \\ &= \mathbf{i}(-3 \cdot 2 - 1 \cdot 9) - \mathbf{j}(3 \cdot 2 - 1 \cdot 4) + \mathbf{k}(3 \cdot 9 + 3 \cdot 4) \\ &= -15\mathbf{i} - 2\mathbf{j} + 39\mathbf{k} \end{aligned}$$

Angular velocity & its relation with linear velocity

Every particle of a rotating body moves in a circle. Angular displacement of a given particle about its centre in unit time is defined as angular velocity.



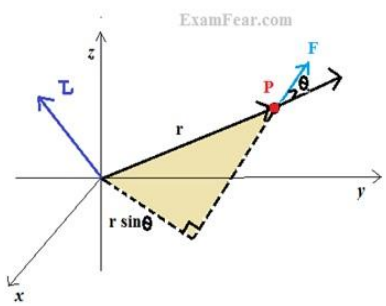
- Average angular velocity = ΔΘ/Δt
- Instantaneous angular velocity, ω = dΘ / dt
- $v = w\,r$, where v – linear velocity of particle moving in a circle of radius r
- All parts of a moving body have the same angular velocity in pure rotation motion.
- Angular velocity, ω, is a vector quantity
- If you curl your fingers of right hand in the sense of rotation, thumb will give direction of angular velocity.



- $v = \omega \times r$
- ANGULAR ACCELERATION is given by rate of change of angular velocity with respect to time.
- $\alpha = d\omega / dt$

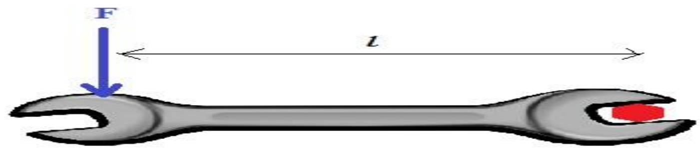
Torque & Angular Momentum

- The rotational analogue of force is moment of force (Torque).
- If a force acts on a single particle at a point P whose position with respect to the origin O is given by the position vector r the moment of the force acting on the particle with respect to the origin O is defined as the vector product $\tau = r \times F = rF \sin\Theta$



- Torque is vector quantity.
- The moment of a force vanishes if either
- The magnitude of the force is zero, or
- The line of action of the force (r sinΘ) passes through the axis.

Example: Determine the torque on a bolt, if you are pulling with a force F directed perpendicular to a wrench of length l cm?
Solution: $t = r \times F = rF \sin\Theta$
In this case $\Theta = 90^o$



- The quantity angular momentum is the rotational analogue of linear momentum.
- It could also be referred to as moment of (linear) momentum.
- l = r × p
- Rotational analogue of Newton's second law for the translational motion of a single particle: dl/st = τ

Torque and angular momentum of system of particles:

- The total angular momentum of a system of particles about a given point is addition of the angular momenta of individual particles added vectorially.

$$L = l_1 + l_2 + \cdots + l_n = \sum_1^n l_i \qquad \text{Where } l_i = r_i \, x \, p_i$$

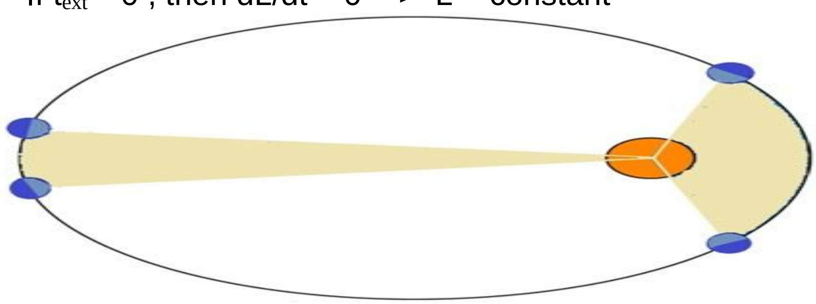
- Similarly for total torque on a system of particles is addition of the torque on an individual particle added vectorially.

$$\tau = \sum_1^n r_i \times F_i = \tau_{ext} + \tau_{int}$$

-
- The torque resulting from internal forces is zero , due to
- Newton's third law i.e. these forces are equal and opposite.
- These forces act on the line joining any two particles
- The time rate of the total angular momentum of a system of particles about a point is equal to the sum of the external torques acting on the system taken about the same point. $\frac{dL}{dt} = \tau_{ext}$

Conservation of Angular Momentum

- if the total external torque on a system of particles is zero, then the total angular momentum of the system is conserved
- If $t_{ext} = 0$, then dL/dt = 0 => L = constant



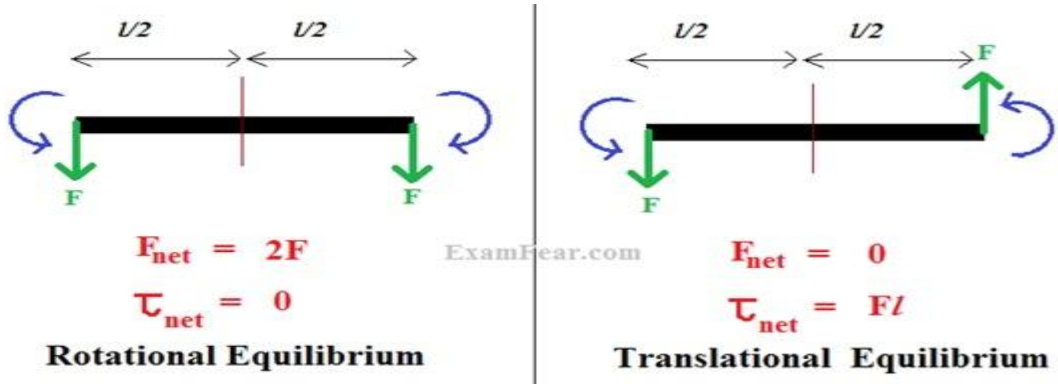
Angualar Momentum of planet is conserved. L = mvr = constant.
When close to Sun their speed will slow down.

Equilibrium of Rigid Body

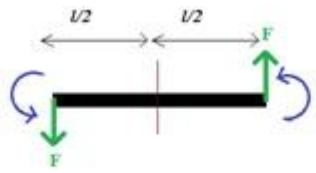
- A force changes the translational state of the motion of the rigid body, i.e. it changes its total linear momentum.
- A torque changes the rotational state of motion of the rigid body, i.e. it changes the total angular momentum of the body

Note: Unless stated otherwise, we shall deal with only external forces and torques.

- A rigid body is said to be in mechanical equilibrium, if both its linear momentum and angular momentum are not changing with time. This means
- Total force should be zero => Translational Equilibrium
- Total torque should be zero => Rotational Equilibrium

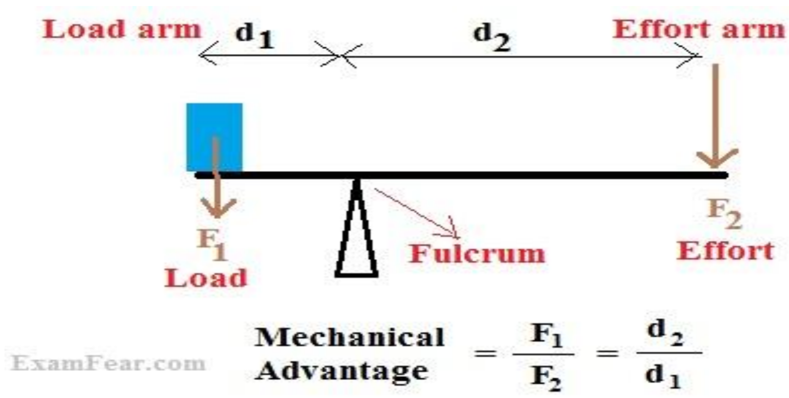


- A pair of equal and opposite forces with different lines of action is known as a couple or torque. A couple produces rotation without translation.



- When you open the lid of a jar , you apply couple on it

- An ideal lever is essentially a light rod pivoted at a point along its length. This point is called the fulcrum
- The lever is a system in mechanical equilibrium.



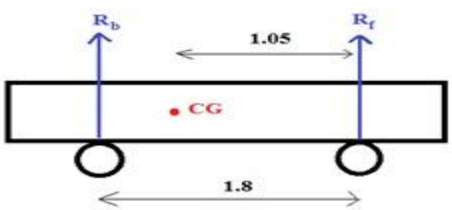
- Mechanical advantage greater than one means that a small effort can be used to lift a large load.

Centre of Gravity

- The centre of gravity of a body is that point where the total gravitational torque on the body is zero.

- The centre of gravity of the body coincides with the centre of mass in uniform gravity or gravity-free space.
- If g varies from part to part of the body, then the centre of gravity and centre of mass will not coincide.

Example: A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.



Solution: Equating forces in vertical direction (translational equilibrium) - $R_f + R_b = mg$ - (1)

Equating the torques at CG (Rotational equilibrium) - $R_f\,(1.05) = R_b\,(1.8-1.05)$ - (2)
Note: We have chosen CG as the torque of gravitational forces is zero at this point.
Solving Eq^n (1)&(2) we can find R_f & R_b .

Moment of Inertia

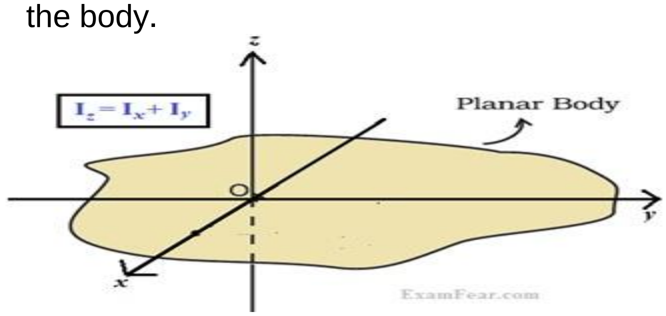
- Moment of inertia (I) is analogue of mass in rotational motion.
- $$I = \sum_{i=1}^{n} m_i r_i^2$$
- Moment of inertia about a given axis of rotation resists a change in its rotational motion; it can be regarded as a measure of rotational inertia of the body.
- It is a measure of the way in which different parts of the body are distributed at different distances from the axis.
- the moment of inertia of a rigid body depends on
- The mass of the body,

- Its shape and size
- Distribution of mass about the axis of rotation
- The position and orientation of the axis of rotation.

- The radius of gyration of a body about an axis may be defined as the distance from the axis of a mass point whose mass is equal to the mass of the whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis.
- I = M k^2, where k is radius of gyration.

Theorem of perpendicular axis

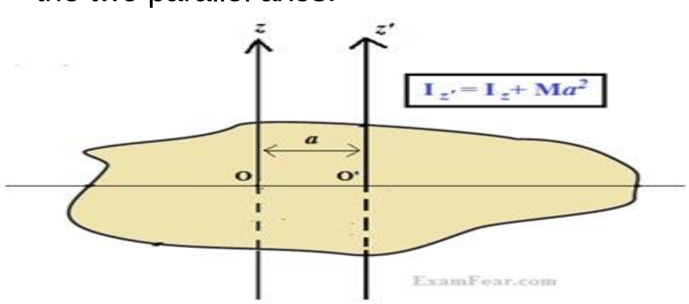
- Perpendicular Axis Theorem: The moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.



- Applicable only to planar bodies.

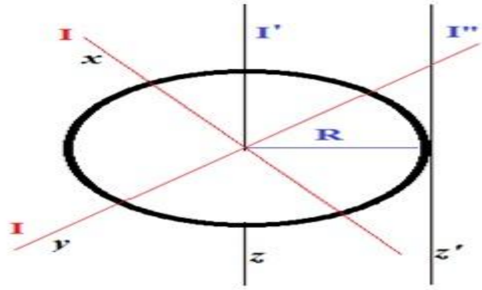
Theorem of parallel axis

- Parallel Axis Theorem: The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.

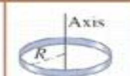
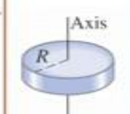
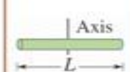
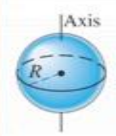


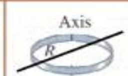
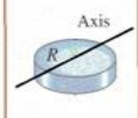
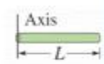
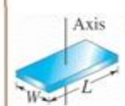
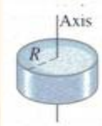
- This theorem is applicable to a body of any shape.

Example: Given the moment of inertia of a disc of mass M and radius R about any of its diameters to be $MR^2/4$, find its moment of inertia about an axis normal to the disc and passing through a point on its edge
Solution:



We can apply Perpendicular axis theorem here on x axis & y axis and get I', moment of inertia in z axis.
$I_z = I_x + I_y$, now as because of symmetry I_x & I_y are same so $I_z = I' = 2I = Mr^2/2$
Now we can apply parallel axis theorem to find I''.
$I'' = I' + MR^2 = 3/2(MR^2)$

Body	Axis	Figure	Moment of Inertia
Thin Circular Ring, Radius R	Perpendicular to plane,at center		MR^2
Circular disc radius R	Perpendicular to disc,at center		$\frac{1}{2}MR^2$
Thin Rod Length L	Perpendicular to rod, at center		$\frac{1}{12}ML^2$
Solid Sphere radius R	Diameter		$\frac{2}{5}MR^2$
Hollow Cylinder Radius,R	Axis of Cylinder		MR^2

Body	Axis	Figure	Moment of Inertia
Thin Circular Ring, Radius R	Diameter		$\frac{1}{2}MR^2$
Circular disc radius R	Diameter		$\frac{1}{4}MR^2$
Thin Rod Length L	Perpendicular to rod, at end point		$\frac{1}{3}ML^2$
Plate Length L, Width W	Perpendicular to plate, at center		$\frac{1}{12}M(L^2+W^2)$
Solid Cylinder Radius, R	Axis of Cylinder		$\frac{1}{2}MR^2$

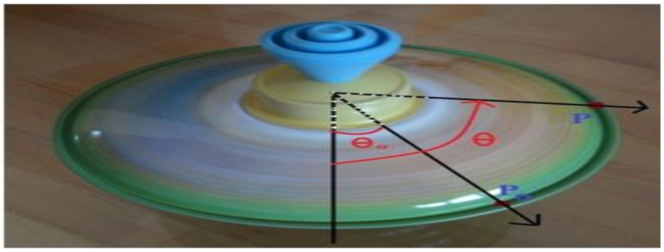
Kinematics of Rotational Motion about a Fixed Axis

- We can derive equation of motion similar to translational motion

Translational Motion		Rotational Motion
$v = v_0 + at$	$\longrightarrow$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\longrightarrow$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2ax$	$\longrightarrow$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Where

- θ_0 = initial angular displacement of the rotating body
- ω_0 = initial angular velocity of the body.
- $\propto$ = angular acceleration, which is constant is this section.



Example:A merry-go-round is accelerated uniformly from rest to an angular velocity of 5 rad/s in a period of 10 seconds. How many times does the merry-go-round make a complete revolution in this time?

Solution: We will use the first eq^n $\omega = \omega_0 + \propto t$

$\propto = 0.5$ rad/s^2

Now we will found the angular displacement by equation: $\theta - \theta_0 = \omega_0 t + \propto t^2/2$

$\Delta\theta = 25$ rad $= 25/2\pi$ rev ~ 4 revolutions.

Dynamics of Rotational Motion about a Fixed Axis

- Only those components of torques, which are along the direction of the fixed axis, need to be considered because the component of the torque perpendicular to the axis of rotation will tend to turn the axis from its position.
- This means
- We need to consider only those forces that lie in planes perpendicular to the axis. Forces which are parallel to the axis will give torques perpendicular to the axis.
- We need to consider only those components of the position vectors which are perpendicular to the axis. Components of position vectors along the axis will result in torques perpendicular to the axis

- Work done by torque is given by: $dW = \tau\, d\theta$
- Power (instantaneous) is given by: $P = \frac{dW}{dt} = \tau\, \omega$
- Kinetic Energy is given by: $K = \frac{1}{2} I\, \omega^2$
- The rate of increase of kinetic energy is $\frac{d}{dt}\left(\frac{I\omega^2}{2}\right) = I\omega \frac{d\omega}{dt}$
  (This is considering the moment of inertia does not change with time.)

- Since $\alpha = \frac{d\omega}{dt}$, $\frac{d}{dt}\left(\frac{I\omega^2}{2}\right) = I\omega\alpha$
- We know that Work Done is equal to Change in Kinetic Energy, $\tau = I\,\alpha$
- The angular acceleration is directly proportional to the applied torque and is inversely proportional to the moment of inertia of the body.
- $\tau = I\,\alpha$ can be called as Newton's second law for rotation about a fixed axis.

	Linear Motion	Rotational Motion about a Fixed Axis
1	Displacement x	Angular displacement θ
2	Velocity $v = dx/dt$	Angular velocity $\omega = d\theta/dt$
3	Acceleration $a = dv/dt$	Angular acceleration $\alpha = d\omega/dt$
4	Mass M	Moment of inertia I
5	Force $F = Ma$	Torque $\tau = I\,\alpha$
6	Work $dW = F\,ds$	Work $W = \tau\,d\theta$
7	Kinetic energy $K = Mv^2/2$	Kinetic energy $K = I\omega^2/2$
8	Power $P = F\,v$	Power $P = \tau\omega$
9	Linear momentum $p = Mv$	Angular momentum $L = I\omega$

Example - Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?

Solution: We know that t = I∝, since torque is equal so

$(I\propto)_{cylinder} = (I\propto)_{sphere}$

$I_{sphere} = 2/5\ MR^2$ & $I_{cylinder} = MR^2$

$\propto_{cylinder} < \propto_{sphere}$

Angular Momentum In Case of Rotation about a Fixed Axis

- We know that L = r x p
  L = r x (mv)
  Now, v = ωr => $L = mr^2\,\omega$
- L = Iω

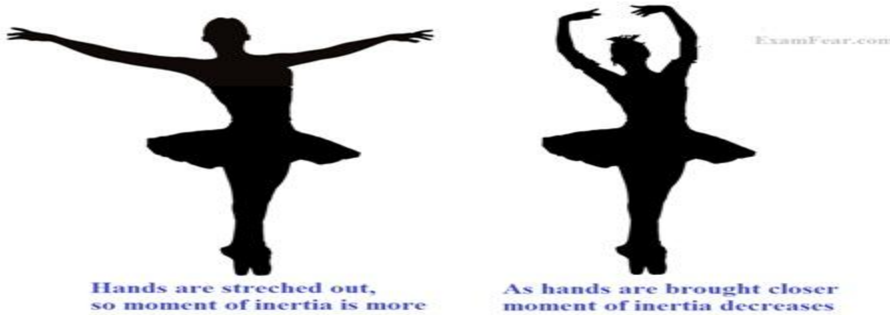
Conservation of angular momentum

- We know that, $\frac{dL}{dt} = \frac{d}{dt}(I\omega) = \tau$
- If $\tau_{ext} = 0$ then Iω = constant

Example:

- When a person is rotating with hands stretched, so the moment of inertia will be more because distribution of mass is far from the axis of rotation.
- As the person brings his arms close to body, moment of inertia decreases because the mass is now distributed close to axis.
- In this situation no external torque is applied, means angular momentum is conserved.
- Iω = constant.

- As I decrease, angular velocity ω



Example - A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value? Assume that the turntable rotates without friction.
Solution: When the child folds his hand back, there is no external torque involved, so the angular momentum is conserved in this case.

We can apply the equation

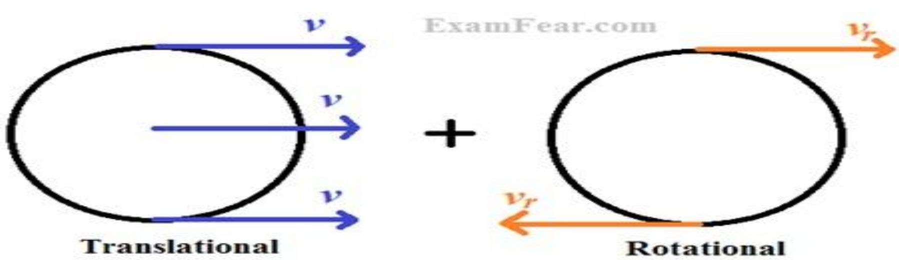
$I\omega$ = constant.

$$I\omega = \left(\frac{2}{5}I\right)\omega'$$

Hence we can find ω'.

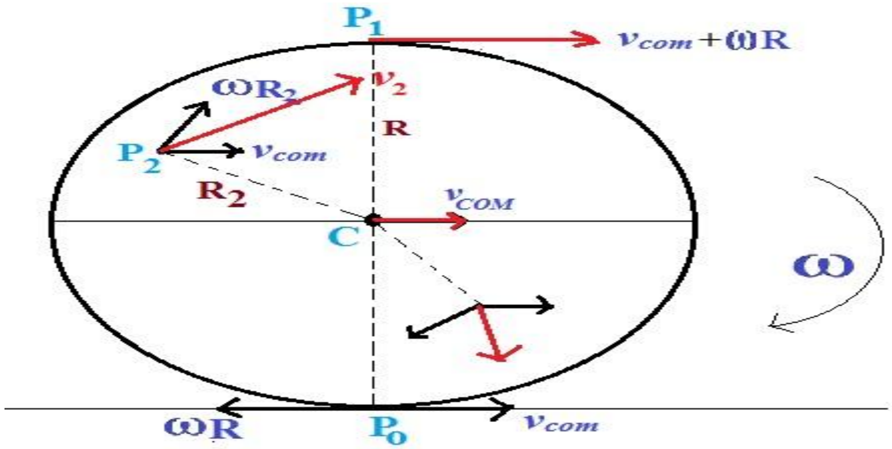
Rolling motion

Rolling motion is a combination of rotation and translation.



- All the particles on a rolling body have two kinds of velocity

1. Translational, which is velocity of COM.
2. Linear velocity on account of rotational motion.



- Here in the figure we can see that every point have two velocities, one in the direction of velocity of COM and other perpendicular to the line joining centre and the point.
- Point P_o have opposite velocities , and if condition of no-slipping is there then it must have zero velocity, so $V_{com} = \omega R$
- At point P_1 both the velocities add up.
- At any other point, add both the velocities vectorially to get the resultant, which are shown for some of the cases in red color in figure.
- The line passing through P_O and parallel to w is called the instantaneous axis of rotation.
- The point P_O is instantaneously at rest.
- Kinetic Energy of Rolling Motion
- $KE_{rolling} = KE_{translation} + KE_{rotation}$

$KE = \frac{1}{2} I\omega^2 + \frac{1}{2} m v_{com}^2$

Substituting $I = mk^2$ (where k is radius of gyration) and $v_{com} = R\,\omega$

We get

$$KE = \frac{1}{2} mv_{com}^2 \left(1 + \frac{k^2}{R^2}\right)$$

Example - A hoop of radius 2 m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20 cm/s. How much work has to be done to stop it?

Solution: The energy required to stop the hoop will be equal to its kinetic energy. (WD=ΔKE)

Now, we know that $KE = \frac{1}{2} I\omega^2 + \frac{1}{2} m v_{com}^2$

And from the condition of rolling $v_{COM} = \omega R$.

Now we know every value required for calculation of KE.

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